Special topics

1. Taylor series (expansion)
2. Fourier Series
3. Fourier transformation
4. Gaussian Beam
5. Determining the Gaussian laser Beam characteristics
6. Periodic optical systems
7. Resonator stability
8. Nonlinear susceptibility
9. Kerr effect
10. Two-photon absorption
11. Second harmonic, sum and difference frequency generation
12. Third harmonic generation
Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. If the Taylor series is centered at zero, then that series is also called a Maclaurin series.

\[
f(x) = f(a) + \frac{f(x)^{(1)}|_{x=a}}{1!}(x-a) + \frac{f(x)^{(2)}|_{x=a}}{2!}(x-a)^2 + \frac{f(x)^{(3)}|_{x=a}}{3!}(x-a)^3 + \ldots
\]

\[
f(x) = \sum_{n=0}^{\infty} \frac{f(x)^{(n)}|_{x=a}}{n!}(x-a)^n
\]

\(f(x)^{(n)}\) is the nth order derivative of function \(f(x)\)

The derivative of order zero of \(f(x)\) is defined to be \(f(x)\) itself and \((x-a)^0\) and \(0!\) are both defined to be 1.
Examples:

Calculate the Taylor expansion of \( f(x) = e^x \) at \( x = 0 \)

\[
f(0) = e^0 = 1
\]
\[
f(x)^{(1)}\bigg|_{x=0} = e^x\bigg|_{x=0} = e^0 = 1
\]
\[
f(x)^{(2)}\bigg|_{x=0} = e^x\bigg|_{x=0} = e^0 = 1
\]
\[
f(x)^{(3)}\bigg|_{x=0} = e^x\bigg|_{x=0} = e^0 = 1
\]

\[
f(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \frac{1}{6!} x^6 + ...
\]
Calculate the Taylor expansion of the following functions at $x = 0$

\[ f(x) = \frac{1}{1-x} \]
\[ f(x) = \frac{1}{1+x} \]
\[ f(x) = \sin(x) \]
\[ f(x) = \cos(x) \]
\[ f(x) = e^{-x^2} \]
\[ f(x) = \sqrt{1 + x^2} \]
\[ f(x) = \sqrt{1 - x^2} \]
Fourier series is a periodic function composed of harmonically related sinusoids, combined by a weighted summation.

When a number of harmonic waves of the same frequency are added together, even though they differ in amplitude and phase, the result is again a harmonic wave of the given frequency, as shown in Chapter 9. If the superposed waves differ in frequency as well, the result is periodic but anharmonic and may assume an arbitrary shape, such as that shown in Figure 12-1. An infinite variety of shapes may be synthesized in this way. The inverse process of decomposition of a given waveform into its harmonic components is called Fourier analysis.

Figure 12-1
If Eq. (1) is multiplied by \( dt \) and integrated over one period \( T \), the sine and cosine integrals vanish, and the result is

\[
f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos m\omega t + \sum_{m=1}^{\infty} b_m \sin m\omega t \quad (1)
\]

\( m \) takes on integral values and \( \omega = 2\pi f = 2\pi / T \), where \( T \) is the period of the arbitrary \( f(t) \). The sine and cosine terms can be interpreted as harmonic waves with amplitudes of \( b_m \) and \( a_m \), respectively, and frequencies of \( m\omega \). The magnitudes of the coefficients or amplitudes determine the contribution each harmonic wave makes to the resultant anharmonic waveform.

If Eq. (1) is multiplied by \( dt \) and integrated over one period \( T \), the sine and cosine integrals vanish, and the result is

\[
a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \, dt \quad (2)
\]
If Eq. (1) is multiplied throughout instead by $\cos(n\omega t) \, dt$, where $n$ is any integer, and then integrated over a period, the only nonvanishing integral on the right side is the one including the coefficient $a_m$ and one finds

$$a_m = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos m\omega t \, dt \quad (3)$$

Similarly, multiplying Eq. (1) by $\sin(n\omega t) \, dt$ and integrating gives

$$b_m = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin m\omega t \, dt \quad (4)$$

Thus, once $f(t)$ is specified, each of the coefficients $a_0$, $a_m$, and $b_m$ can be calculated, and the analysis is complete.
A train of rectangular pulses, has a pulse width equal to $1/4$ of the pulse period. Show that the 4th, 8th 12th etc. harmonics are missing.

\[
A_n = \frac{2}{P} \int_{-P/8}^{P/8} h \cos \left( \frac{2\pi nx}{P} \right) dx \\
= \left( \frac{h}{\pi n} \right) 2 \sin \left( \frac{2\pi n}{P} \cdot \frac{P}{8} \right) \\
= \left( \frac{h}{2} \right) \text{sinc} \left( \frac{\pi n}{4} \right)
\]

so that $A_n = 0$ if $n = 4, 8, 12, \ldots$
Find the sine-amplitude of a saw-tooth waveform as in Fig.

By choosing the origin half way up one of the teeth, the function is clearly made antisymmetrical, so that there are no cosine amplitudes.

\[
B_n = \frac{2}{P} \int_{-P/2}^{P/2} 2 \frac{xh}{P} \sin \left( \frac{2\pi nx}{P} \right) dx
\]

\[
= 4 \frac{h}{P^2} \left[ -x \cos \left( \frac{2\pi nx}{P} \right) \frac{P}{2\pi n} + \frac{P^2}{4\pi^2 n^2} \sin \left( \frac{2\pi nx}{P} \right) \right]_{-P/2}^{P/2}
\]

\[
= (-2h/\pi n) \cos \pi n \quad \text{since} \quad \sin \pi n = 0
\]

so that

\[
B_0 = 0
\]

\[
B_n = (-1)^{n+1} (2h/\pi n), \quad n \neq 0
\]

As a matter of interest, it is worth while calculating the sine-amplitudes when the origin is taken at the tip of a tooth, to see how changing the position of the origin changes the amplitudes. It is also worth while doing the calculation for a similar wave, with negative-going slopes instead of positive.
With the help of Euler's equation, the Fourier series given in general by Eq. (1), involving as it does both \textit{sine} and \textit{cosine} terms, can be expressed in complex notation using exponential functions. The result is

\[ f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{-i\omega t} \]  \hspace{1cm} (5)

where now the coefficients

\[ c_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{i\omega t} \, dt \]  \hspace{1cm} (6)

In cases where we wish instead to represent a nonperiodic function (cleverly interpreted mathematically as a periodic function whose period \( T \) approaches infinity), it is possible to generalize the Fourier series to a \textit{Fourier integral}. For example, a single pulse is a nonperiodic function but can be interpreted as a periodic function whose period extends from \( t = -\infty \) to \( t = +\infty \). It can be shown that the discrete Fourier series now becomes an integral given by
In Figure below, a sample discrete set of coefficients, as might be calculated from Eq. (6), is shown together with a continuous distribution approximated by the coefficients, such as might result from Eq. (8).

\[ f(t) = \int_{-\infty}^{+\infty} g(\omega)e^{-i\omega t} \, d\omega \]  

where the coefficient

\[ g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t)e^{i\omega t} \, dt \]
It should be pointed out that if the function to be represented is a function of spatial position $x$ with period $L$, say, rather than of time $t$ with period $T$, then in Eqs. (1) through (8) $T$ should be replaced by $L$ and the temporal frequency $\omega = 2\pi/T$ should be replaced by the spatial frequency, $k = 2\pi/\lambda$. For example, the Fourier transforms in Eqs. (7) and (8) become:

\[
f(x) = \int_{-\infty}^{+\infty} g(k) e^{-ikx} \, dk \quad (9)
\]

\[
g(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{ikx} \, dx \quad (10)
\]
The ‘top-hat’ function

\[ \Pi_a(x) = \begin{cases} 0, & -\infty < x < -a/2 \\ 1, & -a/2 < x < a/2 \\ 0, & a/2 < x < \infty \end{cases} \]

Its Fourier pair is obtained by integration:

\[
\Phi(p) = \int_{-\infty}^{\infty} \Pi_a(x)e^{2\pi ipx} \, dx
\]

\[
= \int_{-a/2}^{a/2} e^{2\pi ipx} \, dx
\]

\[
= \frac{1}{2\pi ip} [e^{\pi ipa} - e^{-\pi ipa}]
\]

\[
= a \left\{ \frac{\sin \pi pa}{\pi pa} \right\}
\]

\[= a \cdot \text{sinc}(\pi pa)\]
sinc(x) = \sin x / x

Has the value unity at x = 0, and has zeros whenever x = n\pi. The function sinc(\pi pa) above, the most common form, has zeros when p = 1/a, 2/a, 3/a, \ldots

The top-hat function and its transform, the sinc-function.
The Gaussian function

Suppose \( G(x) = e^{-x^2/a^2} \)

\( a \) is the ‘width parameter’ of the function, and the full width at half maximum (FWHM) is 1.386\( a \).

and (what every scientist should know!): \( \int_{-\infty}^{\infty} e^{-x^2/a^2} \, dx = a \sqrt{\pi} \)

Its Fourier transform is \( g(p) \), given by:

\[
g(p) = \int_{-\infty}^{\infty} e^{-x^2/a^2} e^{2\pi ipx} \, dx
\]

The exponent can be rewritten (by ‘completing the square’) as

\[-(x/a - \pi ipa)^2 - \pi^2 p^2 a^2\]

and then

\[
g(p) = e^{-\pi^2 p^2 a^2} \int_{-\infty}^{\infty} e^{-(x/a - \pi ipa)^2} \, dx
\]

put \( x/a - \pi ipa = z \), so that \( dx = adz \). Then:

\[
g(p) = ae^{-\pi^2 p^2 a^2} \int_{-\infty}^{\infty} e^{-z^2} \, dz
\]

\[= a \sqrt{\pi} e^{-\pi^2 a^2 p^2} \]

The Gaussian function and its transform, another Gaussian with full width at half maximum inversely proportional to that of its Fourier pair.
so that \( g(p) \) is another Gaussian function, with width parameter \( 1/\pi a \).

Notice that, the wider the original Gaussian, the narrower will be its Fourier pair.

Notice too, that the value at \( p = 0 \) of the Fourier pair is equal to the area under the original Gaussian.

**The exponential decay**

This, in physics is generally the positive part of the function \( e^{-x/a} \). It is asymmetric, so its Fourier transform is complex:

\[
\Phi(p) = \int_0^\infty e^{-x/a} e^{2\pi i px} \, dx
\]

\[
= \left[ \frac{e^{2\pi i}px - x/a}{2\pi ip - 1/a} \right]_0^\infty = -\frac{1}{2\pi ip - 1/a}
\]
This is a bell-shaped curve, similar in appearance to a Gaussian curve, and is known as a Lorentz profile. It has a FWHM $= 1/\pi a$.

It is the shape found in spectrum lines when they are observed at very low pressure, when collisions between emitting particles are infrequent compared with the transition probability. If the line profile is taken as a function of frequency, $I(\nu)$, the FWHM, $\Delta \nu$ is related to the ‘Lifetime of the Excited State’, the reciprocal of the transition probability in the atom which undergoes the transition. In this example, $a$ and $x$ obviously have dimensions of time. Looked at classically, the emitting particle is behaving like a damped harmonic oscillator radiating power at an exponentially decreasing rate. Quantum mechanics yields the same equation through perturbation theory.

This has the following properties:

**The Dirac ‘delta-function’**

\[
\delta(x) = 0 \text{ unless } x = 0
\]

\[
\delta(0) = \infty
\]

\[
\int_{-\infty}^{\infty} \delta(x)dx = 1
\]
The exponential decay $e^{-|x|/a}$ and its Fourier transform.

When $a$ becomes smaller the width of the exponential decay becomes narrower. Therefore, the Dirac delta function can be regarded as the exponential decay function as its width $a$ goes toward zero.

since it is unbounded at $x = 0$. It can be regarded crudely as the limiting case of a top-hat function $(1/a)\Pi_a(x)$ as $a \to 0$. It becomes narrower and higher, and its area, which we shall refer to as its *amplitude* is always equal to unity. Its Fourier transform is $\text{sinc}(\pi pa)$ and as $a \to 0$, $\text{sinc}(\pi pa)$ stretches and in the limit is a straight line at unit height above the $x$—axis. In other words,
The Fourier transform of a $\delta$-function is unity

and we write:

$$\delta(x) \equiv 1$$

Alternatively, and more accurately, it is the limiting case of a Gaussian function of unit area as it gets narrower and higher. Its Fourier transform then is another Gaussian of unit height, getting broader and broader until in the limit it is a straight line at unit height above the axis.

The following useful properties of the $\delta$-function should be memorized. They are:

$$\delta(x - a) = 0 \text{ unless } x = a$$

The so-called ‘shift theorem’:

$$\int_{-\infty}^{\infty} f(x)\delta(x - a)\,dx = f(a)$$

where the product under the integral sign is zero except at $x = a$ where, on integration, the $\delta$-function has the amplitude $f(a)$. 
It is then easy to show, using this shift theorem, that for positive values of $a$, $b$, $c$ and $d$:

$$\delta(x/a - 1) = a\delta(x - a)$$

$$\delta(a/b - c/d) = ac\delta(ad - bc)$$

$$= bd\delta(ad - bc)$$

$$\delta(ax) = (1/a)\delta(x)$$

And another important consequence of the shift theorem is:

$$\int_{-\infty}^{\infty} e^{2\pi ipx} \delta(x - a) \, dx = e^{2\pi ipa}$$

so that we can write:

$$\delta(x - a) \Leftrightarrow e^{2\pi ipa}$$

$$\delta(mx - a) \Leftrightarrow (1/m)e^{2\pi ipa/m}$$
\[ F_1(x) \Leftrightarrow \Phi_1(p); F_2(x) \Leftrightarrow \Phi_2(p) \]

where ‘\( \Leftrightarrow \)’ implies that \( F_1 \) and \( \Phi_1 \) are a Fourier pair.

\[ F_1(x) + F_2(x) \Leftrightarrow \Phi_1(p) + \Phi_2(p) \]

\[ F_1(x + a) \Leftrightarrow \Phi_1(p)e^{2\pi ipa} \]

\[ F_1(x - a) \Leftrightarrow \Phi_1(p)e^{-2\pi ipa} \]

\[ F_1(x - a) + F_1(x + a) \Leftrightarrow 2\Phi_1(p) \cos 2\pi pa \]
A pair of δ-functions and its transform.
If two $\delta$-functions are equally disposed on either side of the origin, the Fourier transform is a cosine wave:

$$\delta(x - a) + \delta(x + a) \Rightarrow e^{2\pi ipa} + e^{-2\pi ipa}$$

$$= 2 \cos(2\pi pa)$$

**The Dirac comb**

This is an infinite set of equally-spaced $\delta$-functions, usually denoted by the Cyrillic letter $\mathcal{III}$ (Shah). Formally, we write:

$$\mathcal{III}_a(x) = \sum_{n=-\infty}^{\infty} \delta(x - na)$$
It is useful because it allows us to include Fourier series in the general theory of Fourier transforms. For example, the convolution (to be described later) of $\Pi_a(x)$ and $(1/b)\Pi_b(x)$ (where $b < a$) is a square wave similar to that in the earlier example, of period $a$ and width $b$, and with unit area in each rectangle. The Fourier transform is then a Dirac comb, with ‘teeth’ of height $a_m$ spaced at intervals $1/a$. The $a_m$ are of course the coefficients in the series.

If the square wave is allowed to become infinitesimally wide and infinitely high so that the area under each rectangle remains unity, then the coefficients $a_m$ will all become the same height, $1/a$. In other words, the Fourier transform of a Dirac comb is another Dirac comb:

$$\Pi_a(x) \Leftrightarrow \frac{1}{a} \Pi_{\frac{1}{a}}(p)$$

and again notice that the period in $p$-space is the reciprocal of the period in $x$-space.

This is not a formal demonstration of the Fourier transform of a Dirac comb. A rigorous proof is much more elaborate, but is unnecessary here.
The spectral resolution of an infinitely long sinusoidal wave is extremely simple: It is one term of the Fourier series, the term corresponding to the actual frequency of the wave. In this case, all other coefficients vanish. Sinusoidal waves without a beginning or an end are, however, mathematical idealizations. In practice, the wave is turned on and off at finite times. The result is a wave train of finite length. Fourier analysis of such a wave train must regard it as a nonperiodic function. Clearly, it cannot be represented by a single sine wave that has no beginning or end. Rather, the various harmonic waves that combine to produce the wave train must be numerous and so selected that they produce exactly the wave train during the time interval it exists and cancel exactly everywhere outside that interval. Evidently, the turning "on" and "off" of the wave adds many other spectral components to that of the temporary wave train itself. The use of the Fourier-transform integrals leads, in fact, to a continuous distribution of frequency components. What we have said here of a finite wave train is also true of any isolated pulse, regardless of its shape. We consider for simplicity the spectral resolution of a pulse that is, while it exists at some point, a harmonic wave. We have placed the origin of the time frame so that the wave train is symmetrical about it. The wave train has a lifetime of $\tau_0$ and a frequency of $\omega_0$. Thus it may be represented by
\[ f(t) = \begin{cases} 
  e^{-i\omega_0 t}, & -\frac{\tau_0}{2} < t < \frac{\tau_0}{2} \\
  0, & \text{elsewhere}
\end{cases} \]

\[ g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{i\omega t} \, dt = \frac{1}{2\pi} \int_{-\tau_0/2}^{+\tau_0/2} e^{i(\omega - \omega_0)t} \, dt \]

Integrating, we have

\[ g(\omega) = \left[ \frac{e^{i(\omega - \omega_0)t}}{2\pi i(\omega - \omega_0)} \right]_{-\tau_0/2}^{+\tau_0/2} \]

\[ g(\omega) = \frac{1}{\pi(\omega - \omega_0)} \left[ e^{i(\omega - \omega_0)\tau_0/2} - e^{-i(\omega - \omega_0)\tau_0/2} \right] \]
\[ e^{ix} - e^{-ix} = 2i \sin x \]

\[
g(\omega) = \frac{\sin \left[ \frac{\tau_0}{2}(\omega - \omega_0) \right]}{\pi (\omega - \omega_0)} = \frac{\tau_0}{2\pi} \left\{ \frac{\sin \left[ \frac{\tau_0}{2}(\omega - \omega_0) \right]}{[(\tau_0/2)(\omega - \omega_0)]} \right\}
\]

\[ \lim_{\omega \to \omega_0} g(\omega) = \frac{\tau_0}{2\pi} \]

Furthermore, the sinc function \((\sin u)/u\) vanishes whenever \(\sin u = 0\), except at \(u = 0\).

In every other case, \(\sin u \neq 0\) for \(u = n\pi, \ n = \pm 1, \pm 2, \ldots\), and so

\[ g(\omega) = 0 \quad \text{when} \quad \omega = \omega_0 \pm \frac{2n\pi}{\tau_0} \]

As \(\omega\) increases (or decreases) from \(\omega_0\) then, \(g(\omega)\) passes periodically through zero.
Thus the sharper or narrower the pulse, the greater is the number of frequencies required to represent it, and so the greater the line width, or $\Delta \lambda$, of the harmonic wave package.
A real laser beam has a unique direction and finite width. A plane wave or a spherical wave both can not be a real laser beam because a plane wave although has a unique direction but does not have finite width and it spreads all over the space. A spherical wave has a finite width but has not a unique direction and it propagates in all directions. Thus, the wave equation has to be solved with paraxial approximation to derive the Gaussian wave representing a finite width light beam propagating in a certain direction.

When the radiation propagates through the free space (where the polarization is zero) the wave equation is given by:

\[ \nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \]
An spherical monochromatic wave can be the solution for the wave equation

\[ E(r, t) = \frac{A}{r} e^{-i k r} e^{-i \omega t} \]

where \( r \) is the radial coordinate as the distance between the origin and the observation point.

Using paraxial approximation in which a small patch of observation plane around \( z \)-axis is considered, the radial coordinate \( r \) can be approximated as follows

\[
 r = \left( z^2 + x^2 + y^2 \right)^{\frac{1}{2}} = z \left( 1 + \frac{x^2 + y^2}{z^2} \right)^{\frac{1}{2}} \cong z \left( 1 + \frac{x^2 + y^2}{2z^2} \right)
\]
Now the spatial part of solution is modified as

\[ E(r) = \frac{A}{z} \exp \left( -i k z - \frac{i k}{2z} \left( x^2 + y^2 \right) \right) \]

It can be easily shown that if the real parameter \( z \) is replaced by the complex parameter \( z + iz_0 \), \( E(r) \) is still a solution for Helmholtz equation but it attains additional remarkable properties. Thus

\[ E(r) = \frac{A e^{k z_0}}{z + iz_0} \exp \left( - \frac{i k (x^2 + y^2)}{2(z + iz_0)} \right) e^{-i k z} \]

\[ E(r) = \frac{A e^{k z_0}}{iz_0 (1 - i \frac{z}{z_0})} \exp \left( - \frac{ik(x^2 + y^2)}{2} \frac{z - iz_0}{z^2 + z_0^2} \right) e^{-i k z} \]
\[ E(r) = \frac{Ae^{kz_0}(1 + i \frac{z}{z_0})}{iz_0(1 + \frac{z^2}{z_0^2})} \exp \left( -ik \frac{z}{2(z^2 + z_0^2)}(x^2 + y^2) \right) \exp \left( \frac{kz_0}{2(z^2 + z_0^2)}(x^2 + y^2) \right) e^{-ikz} \]

\[
(1 + i \frac{z}{z_0}) = (1 + \frac{z^2}{z_0^2})^\frac{1}{2} e^{\tan^{-1}\left(\frac{z}{z_0}\right)} = (1 + \frac{z^2}{z_0^2})^\frac{1}{2} e^{i\phi} \quad \phi(z) = \tan^{-1}\left(\frac{z}{z_0}\right)\]

\[
R(z) = \frac{z^2 + z_0^2}{z} = z + \frac{z_0^2}{z} \]

\[
w(z)^2 = \frac{2}{k} \left( \frac{z^2 + z_0^2}{z_0} \right) = \frac{2z_0}{k} \left(1 + \frac{z^2}{z_0^2}\right) = w_0^2 \left(1 + \frac{z^2}{z_0^2}\right) \quad w_0 = \sqrt{\frac{2z_0}{k}} \]

\[
E(r) = \left( \frac{Ae^{kz_0}}{iz_0} \right) \frac{e^{i\phi}}{\sqrt{1 + \frac{z^2}{z_0^2}}} \exp \left( -ik(x^2 + y^2) \right) \exp \left( -\frac{x^2 + y^2}{w(z)^2} \right) e^{-ikz} \]

\[ R(z) = z + \frac{z_0^2}{z} \]

the wave front curvature of radius

\[ w(z) = w_0 \left( 1 + \frac{z^2}{z_0^2} \right)^{1/2} \]

the beam radius

\[ w_0 = \sqrt{\frac{2z_0}{k}} = \sqrt{\frac{\lambda z_0}{\pi}} \]

beam waist radius

\[ z_0 = \frac{k w_0^2}{2} = \frac{\pi w_0^2}{\lambda} \]

the Rayleigh range (length)

\[ \phi(z) = \tan^{-1}\left( \frac{z}{z_0} \right) \]

the propagation phase variation
the Rayleigh range that refers to a distance from the beam waist where the beam radius grows by a factor of \( \sqrt{2} \) compared to beam waist radius therefore, the spot area increases by a factor of 2 compared to the beam spot at the beam waist.

\( E \) can be representative for a light beam because it has a finite extent that depends on coordinate \( x, y \) and \( z \) and also it has a unidirectional propagation in the positive \( z \) direction.

Since the radial distribution of the electric field and subsequently the intensity pattern obeys a Gaussian function, this traveling electric field represents the electric field of a Gaussian light beam.

\[
E(r, z) = E_0 \frac{W_0}{W(z)} e^{i\phi(z)} e^{-ikz} e^{-i \frac{r^2}{2R(z)}} e^{-\frac{r^2}{W(z)^2}} \quad r^2 = (x^2 + y^2)
\]
The light intensity is always proportional to the square of the modulus of the electric field thus the spatial intensity distribution of a Gaussian beam is characterized by the Gaussian function as follow:

\[ I(r, z) = I_0 \frac{w_0^2}{w(z)^2} e^{-\frac{2r^2}{w(z)^2}} = I(z)e^{-\frac{2r^2}{w(z)^2}} \]

The light power is obtained by integration the intensity over \( r \) in the plane normal to the propagation direction of the incident beam. So

\[ P = \int_0^\infty I(r, z) (2\pi r) \, dr = \int_0^\infty I_0 \frac{w_0^2}{w(z)^2} e^{-\frac{2r^2}{w(z)^2}} (2\pi r) \, dr \]

\[ P = I_0 \frac{w_0^2}{w(z)^2} (2\pi) \left( \frac{-w(z)^2}{4r} \right) e^{-\frac{2r^2}{w(z)^2}} \bigg|_0^\infty = \pi w_0^2 \frac{I_0}{2} \]
The average intensity over the entire laser spot equals to half of the maximum intensity at the center of spot.

\[ P = \pi w_0^2 \frac{I_0}{2} \implies I = \frac{I_0}{2} \]

\[ P = \pi w_0^2 I \]

\[ P = \int_0^\infty I(r, z) (2\pi r) \, dr = \int_0^\infty I(z) e^{-\frac{r^2}{w(z)^2}} (2\pi r) \, dr = \pi w(z)^2 \frac{I(z)}{2} \]

\[ P = \pi w(z)^2 \frac{I(z)}{2} \implies I = \frac{I_0}{2} \]

\[ P = \pi w(z)^2 I(r, z) \]
The power transferred through an aperture of radius \( a \) can also be calculated by integration of the intensity over \( r \) from the origin up to a distance \( a \).

\[
P = \int_0^a 2\pi r \, dr \, I(r, z) = \frac{\pi w_0^2}{2} I_0 \left( 1 - e^{-\frac{2a^2}{w(z)^2}} \right)
\]

The aperture transmittance defined as the quotient of the power transmitted through the aperture and the entire incident power can be written as:

\[
T = \frac{P(r = a)}{P(r \rightarrow \infty)} = \left( 1 - e^{-\frac{2a^2}{w(z)^2}} \right)
\]

where \( a \) is the aperture radius, \( w(z) \) is the beam radius in the aperture plane and \( z \) is the aperture position.
\[ I(r, z) = I_0 \frac{w_0^2}{w(z)^2} e^{-2 \frac{r^2}{w(z)^2}} = I(z) I(r) \]

\[ I(z) = \frac{w_0^2}{w(z)^2} = \frac{w_0^2}{w_0^2 \left(1 + \frac{z^2}{z_0^2}\right)} = \frac{z_0^2}{z^2 + z_0^2}, \quad I(r) = e^{-2 \frac{r^2}{w(z)^2}} \]

Blue line presents the distribution in \( z \) direction and the Red line for \( r \) direction

For case of tight focus where the beam waist radius is very small

For case of wide focus where the beam waist radius is large
The 2D intensity distribution of a Gaussian beam plotted versus the distance from the optical axis.

Intensity distribution of a Gaussian beam in the xz plane. Figure (a) represents a spherical Voxel for a beam waist diameter of 330 nm. Figure (b) demonstrates an elliptical Voxel when the beam diameter at the beam waist diameter is 2 Micron (scale is not the same for these two figures).
At the origin \((z=0)\) the wavefront curvature radius of the light beam \(R(z)\) is infinite representing a plane wave. The radius of the wavefront curvature reaches its minimum value of \(2z_0\) at the position \(z = z_0\). \(2z_0\) is known as the confocal range in analogy with the distance between two mirrors in a laser resonator with confocal configuration. In the confocal configuration the focal points of two identical mirrors are coincident allowing each mirror located at the central point of another mirror that corresponds to a distance of \(2f\) (\(f\) is the focal length of mirrors) between two mirrors which is equal to the curvature radius of the mirrors.
The beam radius plotted versus the distance from the focal plane as the scale of Rayleigh range.

The Rayleigh range is defined as the distance from the beam waist where the beam radius increases by a factor of $\sqrt{2}$ or correspondingly the intensity decreases by a factor of 2.
Waist of the beam

θ  Beam divergence
Beam divergence

1- half angle

\[ \theta = \frac{w(z)}{z} = \frac{w_0 \left(1 + \frac{z^2}{z_0^2}\right)^{1/2}}{z} \]

This angle is defined as the slope of the tangent line to the beam edge far enough from the beam waist.

\[ z \gg z_0 \Rightarrow \theta = \frac{w_0 \left(\frac{z^2}{z_0^2}\right)^{1/2}}{z} = \frac{w_0 \frac{z}{z_0}}{z} = \frac{w_0}{z_0} = \frac{w_0}{\pi w_0^2} = \frac{\lambda}{\pi w_0} \]

2- full angle: is twice the half angle:

\[ \Phi = 2\theta = \frac{2\lambda}{\pi w_0} = \frac{1.27 \lambda}{D} \]

D: the beam waist diameter
Measurement of Gaussian laser beam radius using the knife-edge technique

\[ P_N = \frac{\int_{-\infty}^{x} \int_{-\infty}^{\infty} I(x', y) \, dy \, dx'}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x', y) \, dy \, dx'} , \]
The area of the photodiode is considered to be larger than the area of the laser beam cross section at the detection position; therefore, diffraction effects may be neglected.

\[ I(x, y) = I_0 \exp \left( -\frac{(x - x_0)^2 + (y - y_0)^2}{w^2} \right), \]

\[ P_N = \frac{\int_{-\infty}^{x} \int_{-\infty}^{\infty} I(x', y) dy dx'}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x', y) dy dx'}, \]

\[ \frac{dP_N(x)}{dx} = \frac{1}{\sqrt{\pi}w} \exp \left[ -\frac{(x - x_0)^2}{w^2} \right]. \]
In a resonator the light rays propagate back and forth therefore, a resonator is a periodic optical system. If the ray transformation matrix of the repeatable element is given by the following:

\[
M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

Then the output parameters after \( n \) repeated optical elements can be derived as:

\[
\begin{bmatrix} y_{\text{out}} \\ \theta_{\text{out}} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdots \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} y_{\text{in}} \\ \theta_{\text{in}} \end{bmatrix}
\]
\[
\begin{bmatrix}
\theta_m \\
y_m
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\cdot
\begin{bmatrix}
y_{m-1} \\
\theta_{m-1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_{m+1} \\
\theta_{m+1}
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\cdot
\begin{bmatrix}
y_m \\
\theta_m
\end{bmatrix}
\]

\[
y_m = A y_{m-1} + B \theta_{m-1} \quad (1)
\]

\[
y_{m+1} = A y_m + B \theta_m \quad (2)
\]

\[
\theta_m = C y_{m-1} + D \theta_{m-1} \quad (3)
\]

\[
\theta_{m+1} = C y_m + D \theta_m \quad (4)
\]

\[
(2) \Rightarrow \theta_m = \frac{y_{m+1} - A y_m}{B} \quad (4)
\]

\[
(3) \Rightarrow \theta_{m-1} = \frac{\theta_m - C y_{m-1}}{D} \quad (5)
\]

\[
(4) \rightarrow (5) \Rightarrow \theta_{m-1} = \frac{y_{m+1} - A y_m}{B} - C y_{m-1} = \frac{y_{m+1} - A y_m - B C y_{m-1}}{B D} \quad (6)
\]
(1) \( \Rightarrow y_m = A y_{m-1} + B \frac{y_{m+1} - A y_m - B C y_{m-1}}{B D} \) \hspace{1cm} (7)

\[ D y_m = A D y_{m-1} + y_{m+1} - A y_m - B C y_{m-1} \] \hspace{1cm} (8)

\[ y_{m+1} - (A + D) y_m + (A D - B C) y_{m-1} = 0 \] \hspace{1cm} (9)

The determination of the ray transfer matrix is equal to unity:

\[ (A D - B C) = 1 \] \hspace{1cm} (10)

\[ (10) \rightarrow (9) \Rightarrow y_{m+1} - (A + D) y_m + y_{m-1} = 0 \] \hspace{1cm} (11)

\[ y_{m+1} - 2b y_m + y_{m-1} = 0 \hspace{1cm} (b = \frac{A + D}{2}) \] \hspace{1cm} (12)
In order to solve the Eq. (12) we can assume that:

\[ y_m = y_0 h^m \quad (13) \]

\[ (13) \rightarrow (12) \Rightarrow h^{m+1} - 2bh^m + h^{m-1} = 0 \quad (14) \]

\[ h^2 - 2bh + 1 = 0 \quad (15) \]

\[ h = b \pm \sqrt{b^2 - 1} = b \pm i \sqrt{1 - b^2} \quad (16) \]

\[ |h| = \sqrt{h \times h^*} = \sqrt{(b + i \sqrt{1 - b^2}) \times (b - i \sqrt{1 - b^2})} = \sqrt{b^2 + 1 - b^2} = 1 \]
The beam position, $y$, would oscillate and come back to its previous position if $\Phi$ is real.

\[
h = \cos(\Phi) \pm i \sin(\Phi) = e^{\pm i\Phi} \quad (17)
\]

\[
y_m = y_0 h^m = y_0 e^{\pm im\Phi} \quad (18)
\]

\[
\cos(\Phi) = b \\
\sin(\Phi) = \sqrt{1 - b^2} \\
\tan(\Phi) = \frac{\sqrt{1 - b^2}}{b}
\]
\( \Phi \text{ is real} \Rightarrow b^2 \leq 1 \Rightarrow -1 \leq b \leq 1 \)

**Stability condition:**  
\[-1 \leq \frac{A + D}{2} \leq 1 \quad (19)\]

For the case of real \( \Phi \) the position of beam is a harmonic function of \( m \). That means, in the periodic optical system the characteristics of beam \( (y \text{ and } \alpha) \) will become the same after passing through \( s \) similar repeatable optical element. Mathematically that means:

\[
y_{m+s} = y_m \Rightarrow y_0 e^{\pm i(m+s)\Phi} = y_0 e^{\pm im\Phi} \\
\]

\[
s \Phi = 2\pi \\
\]

\[
s = \frac{2\pi}{\Phi} \\
\]
Example:
A set of similar lenses arranging in a line with equal distance

In this periodic optical system, the light beam travels through free space with length of $d$ and then refracted by a lens. Thus, the repeated optical element consist of two elements: propagating in free space and refraction by a thin lens. Therefore, the ray transfer matrix of the repeated optical system can be written as:

$$
M = \begin{bmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{bmatrix} \times \begin{bmatrix}
1 & d \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
1 & -\frac{1}{f} \\
-\frac{1}{f} & 1 - \frac{1}{f}
\end{bmatrix}
$$
It can be assumed that the light is first refracted by a thin lens and then travels through free space with length of $d$. Thus, the repeated optical element consist of two elements: refraction by a thin lens and propagating in free space. Therefore, the ray transfer matrix of the repeated optical system can be written as:

$$M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{f} & d \\ -\frac{1}{f} & 1 \end{bmatrix}$$

Although $A$ and $D$ are different from previous matrix the term $b = \frac{A + D}{2}$ is the same leading the same results.
Stability criterion:

\[ -1 \leq b \leq 1 \Rightarrow -1 \leq 1 - \frac{d}{2f} \leq 1 \Rightarrow 0 \leq d \leq 4f \]

For: \( d = f \) \( \Rightarrow b = \frac{1}{2} \Rightarrow \Phi = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \)

\[ S = \frac{2\pi}{\Phi} = \frac{2\pi}{\frac{\pi}{3}} = 6 \Rightarrow y_6 = y_0 \]
For: \( d = 2f \Rightarrow b = 0 \Rightarrow \Phi = \cos^{-1}(0) = \frac{\pi}{2} \)

\[
S = \frac{2\pi}{\Phi} = \frac{2\pi}{\frac{\pi}{2}} = 4 \Rightarrow y_4 = y_0
\]
for: \( d = 3f \) \( \Rightarrow \) \( b = -\frac{1}{2} \) \( \Rightarrow \) \( \Phi = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \)

\[
S = \frac{2\pi}{\Phi} = \frac{2\pi}{2\pi} = 3 \quad \Rightarrow \quad y_3 = y_0
\]

\[
y_4 = y_1
\]

\[
y_5 = y_2
\]
for: \( d = 4f \implies b = -1 \implies \Phi = \cos^{-1}(-1) = \pi \)

\[
S = \frac{2\pi}{\Phi} = \frac{2\pi}{\pi} = 2 \implies y_2 = y_0
\]

\[
y_3 = y_1
\]

\[
y_4 = y_2
\]
A resonator is consist of two mirrors standing face to face with distance of $d$. Therefore, the light ray travels two times through free space and reflect two time from mirrors. That means, the optical element which repeats consists of two propagation through free space and reflection from two mirrors.

$$M = \begin{bmatrix}
  1 & 0 \\
  -\frac{2}{R_2} & 1 \\
\end{bmatrix} \times \begin{bmatrix}
  1 & d \\
  0 & 1 \\
\end{bmatrix} \times \begin{bmatrix}
  1 & 0 \\
  -\frac{2}{R_1} & 1 \\
\end{bmatrix} \times \begin{bmatrix}
  1 & d \\
  0 & 1 \\
\end{bmatrix}$$
\[ M = \begin{bmatrix}
1 - \frac{2d}{R_1} & 2d \left(1 - \frac{d}{R_1}\right) \\
\frac{2}{R_1} + \frac{2}{R_2} + \frac{4d}{R_1 R_2} & 1 - \frac{2d}{R_1} - \frac{4d}{R_2} + \frac{4d^2}{R_1 R_2}
\end{bmatrix} \]

Stability criterion:

\[-1 \leq A + D \leq 1\]

\[-1 \leq \frac{1 - \frac{2d}{R_1} + 1 - \frac{2d}{R_1} - \frac{4d}{R_2} + \frac{4d^2}{R_1 R_2}}{2} \leq 1\]

\[-1 \leq 1 - \frac{2d}{R_1} - \frac{2d}{R_2} + \frac{2d^2}{R_1 R_2} \leq 1\]
\[ 0 \leq 2 - \frac{2d}{R_1} - \frac{2d}{R_2} + \frac{2d^2}{R_1 R_2} \leq 2 \]

\[ 0 \leq 1 - \frac{d}{R_1} - \frac{d}{R_2} + \frac{d^2}{R_1 R_2} \leq 1 \]

\[ 0 \leq (1 - \frac{d}{R_1})(1 - \frac{d}{R_2}) \leq 1 \]

\[ 0 \leq g_1 g_2 \leq 1 \]

\[ g_1 = (1 - \frac{d}{R_1}) \quad , \quad g_2 = (1 - \frac{d}{R_2}) \quad , \quad b = \frac{A + D}{2} = 2g_1 g_2 - 1 \]
A $g_1$, $g_2$ stability diagram for a general spherical resonator. The stable region corresponds to shaded parts in the figure; dashed curves correspond to the possible confocal resonators.
\[ \Phi = \cos^{-1}(b) = \cos^{-1}\left(\frac{A + D}{2}\right) = \cos^{-1}(2g_1 g_2 - 1) \]

\[ s = \frac{2\pi}{\Phi} = \frac{2\pi}{\cos^{-1}(g_1 g_2)} = \frac{2\pi}{\cos^{-1}\left(2(1 - \frac{d}{R_1})(1 - \frac{d}{R_2}) - 1\right)} \]

if \( R_1 = R_2 \Rightarrow g_1 = g_2 = 1 - \frac{d}{R} \Rightarrow b = 2g^2 - 1 = 2\left(1 - \frac{d}{R}\right)^2 - 1 \]

Stability criterion: \(-1 \leq b \leq 1 \Rightarrow 0 \leq d \leq 2R \)

\[ s = \frac{2\pi}{\cos^{-1}\left(2(1 - \frac{d}{R})^2 - 1\right)} \]
\[ d = \frac{R}{2} \Rightarrow s = \frac{2\pi}{\cos^{-1}\left(2\left[1 - \frac{R/2}{R}\right]^2 - 1\right)} = \frac{2\pi}{\cos^{-1}\left(-\frac{1}{2}\right)} = \frac{2\pi}{3} = 3 \]

\[ d = R \Rightarrow s = \frac{2\pi}{\cos^{-1}\left(2(1 - \frac{R}{R})^2 - 1\right)} = \frac{2\pi}{\cos^{-1}\left(-1\right)} = \frac{2\pi}{\pi} = 2 \quad \text{Confocal resonator} \]
\[ d = \frac{3R}{2} \implies s = \frac{2\pi}{\cos^{-1}\left(2\left[1 - \frac{3R/2}{R}\right]^2 - 1\right)} = \frac{2\pi}{\cos^{-1}\left(-\frac{1}{2}\right)} = \frac{2\pi}{3} = 3 \]

\[ d = 2R \implies s = \frac{2\pi}{\cos^{-1}\left(2\left[1 - \frac{2R}{R}\right]^2 - 1\right)} = \frac{2\pi}{\cos^{-1}(1)} = \frac{2\pi}{2\pi} = 1 \quad \text{concentric resonator} \]

\[ d = 2R = 4f \]
if $R_1 = \infty \Rightarrow g_1 = 1 \Rightarrow b = 2g - 1 = 2\left(1 - \frac{d}{R}\right) - 1$

Stability criterion: $-1 \leq b \leq 1 \Rightarrow 0 \leq d \leq R$

$$s = \frac{2\pi}{\cos^{-1}\left(2\left(1 - \frac{d}{R}\right) - 1\right)}$$

Flat mirror is mounted at the focal point of the spherical mirror

$$d = \frac{R}{2} \Rightarrow s = \frac{2\pi}{\cos^{-1}\left(2\left(1 - \frac{R/2}{R}\right) - 1\right)} = \frac{2\pi}{\cos^{-1}(0)} = \frac{2\pi}{\pi} = 4$$

Hemiconfocal resonator

$R_1 = \infty, R_2 = 2L$

$g_1g_2 = \frac{1}{2}$
Flat mirror is mounted at the center of the spherical mirror

\[
d = R \implies s = \frac{2\pi}{\cos^{-1}\left(2\left(1 - \frac{R}{d}\right) - 1\right)} = \frac{2\pi}{\cos^{-1}(-1)} = \frac{2\pi}{\pi} = 2
\]

Hemispherical resonator

\[R_1 = \infty, \quad R_2 = L\]

\[g_1g_2 = 0\]
If \( R_1 = \infty \text{ and } R_2 = \infty \Rightarrow g_1 = 1 \text{ and } g_2 = 1 \Rightarrow b = 1 \)

**Stability criterion:** \( 0 \leq d \leq \infty \)

\[
s = \frac{2\pi}{\cos^{-1}(1)} = \frac{2\pi}{2\pi} = 1
\]

A resonator consisting of two parallel plane mirrors is always stable regardless of the length of the resonator.

In such a resonators, the beams parallel to the optical axis will remain within the resonators and back to the same position after one round trip. those beams which deviate from the optical axis will give up the laser medium as loss.
When a medium is placed in an applied external electric field, the electron cloud is deformed leading to create a dipole moment due to electric force arising from Coulomb law.

The induced polarization can be written as a function of applied electric field using the Tylor expansion:

\[ P = \frac{dP}{dE} E + \frac{1}{2} \frac{d^2 P}{dE^2} E^2 + \frac{1}{6} \frac{d^3 P}{dE^3} E^3 + \ldots \]

\[ P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(2)} E^2 + \chi^{(3)} E^3 + \ldots \right) \]

\[ \chi^{(n)} : nth \ order \ susceptibility \]

\[ \chi^{(1)} = \frac{1}{\varepsilon_0} \frac{dP}{dE} \]

\[ \chi^{(2)} = \frac{1}{2 \varepsilon_0} \frac{d^2 P}{dE^2} \]

\[ \chi^{(3)} = \frac{1}{6 \varepsilon_0} \frac{d^3 P}{dE^3} \]
The second order susceptibility \( \chi^{(2)} \) is about 8 order of magnitude smaller than the first order \( \chi^{(1)} \). Thus, for common light sources where the light intensity is low the second and also other terms in polarization expansion is negligible with respect to the first term. Therefore:

\[
P = \varepsilon_0 \chi^{(1)} E
\]

It means that the polarization is a linear function of the applied electric field. The area of optics in which this approximation is held is called linear optics.

When a medium interact with high enough intense light the other terms in Taylor expansion of polarization versus electric field is no longer negligible and thus the polarization is a nonlinear function of applied electric field. This area of optics is called nonlinear optics.
Consider an atom with electrons bound to the nucleus in much the same way as a small mass can be bound to a large mass by a spring. This is the Lorentz model.

The net force applied to the electron in an oscillating external electric field considering a damp factor is:

$$\sum f = -k x - m \Gamma \left( \frac{dx}{dt} \right) - e E(t)$$

$k$ is the spring constant, $m$ is the electron mass, $\Gamma$ is the damping factor, $e$ is the electron electric charge and $E$ is the applied electric field.

The motion of an electron bound to the nucleus is then:

$$m \frac{d^2 x}{dt^2} + m \Gamma \frac{dx}{dt} + m \omega_0^2 x = -e E_0 e^{-i\omega t} \left( \omega_0 = \sqrt{\frac{k}{m}} \right)$$
The oscillation amplitude of electron is then derived as:

\[ x(t) = \frac{eE_0}{m} \frac{1}{(\omega_0^2 - \omega^2) - i\Gamma \omega} e^{-i\omega t} \]

Then the dipole moment of the oscillating electron can be written:

\[ p(t) = e x(t) = \frac{e^2 E_0}{m} \frac{1}{(\omega_0^2 - \omega^2) - i\Gamma \omega} e^{-i\omega t} \]

The polarization in then given by:

\[ P(t) = Np(t) = \frac{Ne^2}{m} \frac{1}{(\omega_0^2 - \omega^2) - i\Gamma \omega} (E_0 e^{-i\omega t}) \]

The polarization oscillates with frequency of \( \omega \) thus the atom radiate only with frequency of \( \omega \). This explain the scattering and reflecting
Now the first order (linear) susceptibility can be derived as:

\[ P(t) = \varepsilon_0 \chi^{(1)} E(t) \]

\[ \chi^{(1)} = \frac{N e^2}{m \varepsilon_0} \frac{1}{(\omega_0^2 - \omega^2) - i\Gamma \omega} \]

From which the linear refractive index and linear absorption coefficient can be calculated.

\[ n = \sqrt{1 + \chi^{(1)}} \]
we extend the Lorentz model by allowing the possibility of a nonlinearity in the restoring force exerted on the electron. The details of the analysis differ depending upon whether or not the medium possesses inversion symmetry.

Non-centro-symmetric Media

the potential energy function is of the form

\[ U = \frac{1}{2} k x^2 + \frac{1}{3} A x^3 \]

Then the restoring force is given by:

\[ F = -k x - A x^2 \]

And then the polarization possesses a nonlinear term as:

\[ P = \varepsilon_0 (\chi^{(1)} E + \chi^{(2)} E^2) \]
\[ \frac{1}{2} k x^2 \]

\[ \frac{1}{2} k x^2 - \frac{1}{3} A x^3 \]

\[ \frac{1}{2} k x^2 + \frac{1}{3} A x^3 \]

\[ P = \varepsilon_0 (\chi^{(1)} E + \chi^{(2)} E^2) \]
\[ P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(2)} E^2 \right) = \varepsilon_0 \left( \chi^{(1)} E_0 \cos(\omega t) + \chi^{(2)} \left( E_0 \cos(\omega t) \right)^2 \right) \]

\[ P = \frac{1}{2} \varepsilon_0 \chi^{(2)} E_0^2 + \varepsilon_0 \chi^{(1)} E_0 \cos(\omega t) + \frac{1}{2} \varepsilon_0 \chi^{(2)} E_0^2 \cos(2\omega t) \]
 Centro-symmetric Media

the potential energy function is of the form which is symmetric under the operation of  $x \rightarrow -x$

$$U = \frac{1}{2} k x^2 + \frac{1}{4} B x^4$$  \hspace{1cm} \text{Anharmonic Oscillator}

Then the restoring force is given by:

$$F = -k x - B x^3$$

For centrosymmetric it can be shown that the first nonlinear term appeared in the polarization is of the third order

$$P = \varepsilon_0 (\chi^{(1)} E + \chi^{(3)} E^3)$$
\[ P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(3)} E^3 \right) = \varepsilon_0 \left( \chi^{(1)} E_0 \cos(\omega t) + \chi^{(3)} \left( E_0 \cos(\omega t) \right)^3 \right) \]

\[ P = \varepsilon_0 \left( \chi^{(1)} + \frac{3}{4} \chi^{(3)} E_0^2 \right) E_0 \cos(\omega t) + \frac{1}{4} \varepsilon_0 \chi^{(3)} E_0^3 \cos(3\omega t) \]
**Kerr Effect**

\[ p = \varepsilon_0 \left( \chi^{(1)} E \cos(\omega t) + \chi^{(3)} (E \cos(\omega t))^3 \right) \]

\[ P = \varepsilon_0 \left( \chi^{(1)} + \frac{3}{4} \chi^{(3)} E_0^2 \right) E_0 \cos(\omega t) + \frac{1}{4} \varepsilon_0 \chi^{(3)} E_0^3 \cos(3\omega t) \]

\[ p_\omega(t) = \varepsilon_0 (\chi^{(1)} + \frac{3}{4} E^2 \chi^{(3)}) E \cos(\omega t) = \varepsilon_0 \chi_{\text{eff}}^{(1)} E \cos(\omega t) \]

**Complex Refractive index:**

\[ n^* = \sqrt{1 + \chi_{\text{eff}}^{(1)}} = \sqrt{1 + \chi^{(1)} + \frac{3}{4} \chi^{(3)} E^2} \]

\[ n^* = \sqrt{\left(1 + \chi_R^{(1)} + \frac{3}{4} \chi_R^{(3)} E^2 \right) + i \left( \chi_I^{(1)} + \frac{3}{4} \chi_I^{(3)} E^2 \right)} \]

Both refractive index and susceptibility are complex
\[ n^* = n_R + i n_I = \sqrt{1 + \chi_R^{(1)} + \frac{3}{4} \chi_R^{(3)} E^2} + i \frac{\chi_I^{(1)} + \frac{3}{4} \chi_I^{(3)} E^2}{2\sqrt{1 + \chi_R^{(1)} + \frac{3}{4} \chi_R^{(3)} E^2}} \]

Refractive Index = Real part of complex refractive index:

\[ n = n_R = \sqrt{1 + \chi_R^{(1)} + \left( \frac{3 \chi_R^{(3)}}{8 \sqrt{1 + \chi_R^{(1)}}} \right) E^2} \]

\[ n_0 = \sqrt{1 + \chi_R^{(1)}} \quad \text{Linear refractive index} \]

\[ n = n_0 + \left( \frac{3 \chi_R^{(3)}}{4 n_0^2 c \varepsilon_0} \right) I \quad \Rightarrow \quad n = n_0 + n_2 I \]

\[ n_2 = \frac{3 \chi_R^{(3)}}{4 n_0^2 c \varepsilon_0} \quad \text{Nonlinear refractive index} \]
Self focusing: $n_2 > 0$
Self focusing: $n_2 > 0$

Nonlinear Sample

Gaussian Input beam

Self-Focused beam

Nonlinear Sample

\[
\Delta \Phi = \frac{2\pi}{\lambda} \Delta n(I) L = \frac{2\pi}{\lambda} n_2 I L
\]
Self-focusing

Lower $n$ \rightarrow shorter optical path \leftarrow Shorter $L$

Higher $n$ \rightarrow Longer optical path \leftarrow longer $L$

$\Delta = nL$

Self-defocusing

Higher $n$ \rightarrow Longer optical path \leftarrow Shorter $L$

Lower $n$ \rightarrow shorter optical path \leftarrow longer $L$

Higher $n$
Two-photon Absorption

Non-degenerate process

Excited state

Virtual state

Ground state

$\omega_1$

$\omega_2$

Degenerate process

Excited state

Virtual state

Ground state

$\omega$

$\omega$

Non-degenerate process

Degenerate process
\[ n_I = \frac{\chi_I^{(1)} + \frac{3}{4} \chi_I^{(3)} E^2}{2 \sqrt{1 + \chi_R^{(1)}}} = \frac{\chi_I^{(1)}}{2 n_0} + \left( \frac{3 \chi_I^{(3)}}{4 n_0^2 c \varepsilon_0} \right) I \]

\[ \alpha(I) = \frac{2 \omega}{c} n_I = \frac{\omega \chi_I^{(1)}}{c n_0} + \left( \frac{3 \omega \chi_I^{(3)}}{2 n_0^2 c^2 \varepsilon_0} \right) I = \alpha_0 + \alpha_2 I \]

\[ \begin{align*}
\alpha_0 &= \frac{\omega \chi_I^{(1)}}{c n_0} & \text{Linear absorption coefficient} \\
\alpha_2 &= \frac{3 \omega \chi_I^{(3)}}{2 n_0^2 c^2 \varepsilon_0} & \text{Two-photon absorption coefficient}
\end{align*} \]
\[
\frac{dI}{dz'} = -\alpha(I) I = -\left(\alpha_0 + \alpha_2 I\right) I
\]

\[
I(z') = \frac{\alpha_0 I(z) e^{-\alpha_0 z'}}{\alpha_0 + \alpha_2 I(z)(1 - e^{-\alpha_0 z'})}
\]

\[
\alpha_2 = 0 \Rightarrow I(z') = I(z) e^{-\alpha_0 z'}
\]

Beer–Lambert law
Nonlinear phenomena and their Applications

- **Higher harmonic generation:**
  - **second harmonic generation (SHG):** in this process two photons of the same frequency are annihilated and one photon at double frequency is created.
\[ P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(2)} E^2 \right) \]

\[ E = E_0 e^{i\omega t} + E_0^* e^{-i\omega t} \quad \text{The electric field oscillate at frequency } \omega \]

\[ P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(2)} (E_0 e^{i\omega t} + E_0^* e^{-i\omega t})^2 \right) \]

\[ P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(2)} (E_0^2 e^{i2\omega t} + E_0^* e^{-i2\omega t} + 2E_0 E_0^*) \right) \]

\[ P = \varepsilon_0 \left( 2|E_0|^2 + \chi^{(1)} E_0 e^{i\omega t} + \chi^{(2)} E_0^2 e^{i2\omega t} \right) = P_0 + P_\omega + P_{2\omega} \]

\[ P_0 = 2\varepsilon_0 |E_0|^2 \quad \text{Optical Rectification. Non-oscillating term of polarization.} \]
\[ \text{Charge separation caused by high intensity light beam} \]

\[ P_\omega = \varepsilon_0 \chi^{(1)} E_0 e^{i\omega t} \quad \text{Oscillating term at frequency } \omega. \text{ This terms is the origin of reemitting leading to normal scattering.} \]

\[ P_{2\omega} = \varepsilon_0 \chi^{(2)} E_0^2 e^{i2\omega t} \quad \text{This term shows the contribution of nonlinear polarization oscillating at frequency } 2\omega \text{ which is the origin of second harmonic generation.} \]
Optical Parametric Amplification

in this process photon at new frequencies can be generated. Devices such as Optical Parametric Oscillator (OPO) and Optical Parametric Amplifier (OPA) work based on this phenomena. When two intense laser beam with different frequencies travel through a special nonlinear crystal, two different processes might be occurred depending on the system alignment

✓ sum frequency generation: frequency of the new generated photon equals to sum of the two incident photons

\[ \omega_{new} = \omega_1 + \omega_2 \]

✓ difference frequency generation: frequency of the new generated photon equals to difference of the two incident photons

\[ \omega_{new} = |\omega_1 - \omega_2| \]
\[ P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(2)} E^2 \right) = P_1 + P_2 \]

\[ E = E_{01} e^{i\omega_1 t} + E_{01}^* e^{-i\omega_1 t} + E_{02} e^{i\omega_2 t} + E_{02}^* e^{-i\omega_2 t} \]

\[ P_2 = \varepsilon_0 \chi^{(2)} \left( E_{01} e^{i\omega_1 t} + E_{01}^* e^{-i\omega_1 t} + E_{02} e^{i\omega_2 t} + E_{02}^* e^{-i\omega_2 t} \right)^2 \]

\[ P_2 = \varepsilon_0 \chi^{(2)} \left( 2E_{01} E_{01}^* + E_{01}^2 e^{2i\omega_1 t} + E_{02}^2 e^{2i\omega_2 t} + E_{01} E_{02} e^{i(\omega_1+\omega_2)t} + E_{01} E_{02}^* e^{i(\omega_1-\omega_2)t} \right) \]

\[ P_2 = P_0 + P_{2\omega_1} + P_{2\omega_2} + P_{(\omega_1+\omega_2)} + P_{(\omega_1-\omega_2)} \]

\[ P_0 = 2\varepsilon_0 \left| E_0 \right|^2 \quad \text{Optical Rectification.} \]

\[ P_{2\omega_1} = \varepsilon_0 \chi^{(2)} E_{01}^2 e^{i2\omega_1 t} \quad \text{Second harmonic generation (SHG).} \]

\[ P_{(\omega_1+\omega_2)} = \varepsilon_0 \chi^{(2)} E_{01} E_{02} e^{i(\omega_1+\omega_2)t} \quad \text{Sum frequency generation (SFG).} \]

\[ P_{(\omega_1-\omega_2)} = \varepsilon_0 \chi^{(2)} E_{01} E_{02} e^{i(\omega_1-\omega_2)t} \quad \text{Difference frequency generation (DFG).} \]
Parametric Down-Conversion (Difference-frequency generation)

Optical Parametric Amplification (OPA)

Optical Parametric Oscillation (OPO)

Optical Parametric Generation (OPG)

mirror

"signal"

"idler"
Some typical applications of sum frequency generation are:

- Generation of red light (→ red lasers), e.g. by mixing the outputs of a 1064-nm Nd:YAG laser and a 1535-nm fiber laser, resulting in an output at 628 nm.
- Generation of ultraviolet light, e.g. by mixing the output of a 1064-nm Nd:YAG laser with frequency-doubled light at 532 nm, resulting in 355-nm UV light:

\[
\omega_{\text{new}} = \omega_1 + \omega_2 \Rightarrow \frac{1}{\lambda_{\text{new}}} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \Rightarrow \lambda_{\text{new}} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}
\]

\[
\lambda_1 = 1064 \text{ nm}, \quad \lambda_2 = 532 \text{ nm}
\]

\[
\lambda_{\text{new}} = \frac{1064 \times 532}{1064 + 532} = \frac{1064}{3} = 355 \text{ nm}
\]

Some typical applications of difference frequency generation are:

- Generation of light around 3.3 μm by mixing 1570 nm from a fiber laser and 1064 nm.
- Generation of light around 4.5 μm by mixing 860 nm from a laser diode and 1064 nm.
Third Harmonic Generation

\[ P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(2)} E^2 + \chi^{(3)} E^3 + \ldots \right) = P^{(1)} + P^{(2)} + P^{(3)} \]

\[ E = E_0 e^{i\omega t} + E_0^* e^{-i\omega t} \text{ Monochromatic light} \]

\[ P^{(3)} = \varepsilon_0 \chi^{(3)} E^3 = \varepsilon_0 \chi^{(3)} \left( E_0 e^{i\omega t} + E_0^* e^{-i\omega t} \right)^3 \]

\[ P^{(3)} = \varepsilon_0 \chi^{(3)} E^3 = \varepsilon_0 \chi^{(3)} \left( E_0^3 e^{i(3\omega)t} + (E_0^*)^3 e^{-i(3\omega)t} + 3|E_0|^2 E_0 e^{i\omega t} + 3|E_0|^2 E_0^* e^{-i\omega t} \right) \]

\[ P^{(3)}_{\omega} = \varepsilon_0 \chi^{(3)} \left( 3|E_0|^2 E_0 e^{i\omega t} \right) \text{ compounent oscillating at } \omega \]

\[ P^{(3)}_{3\omega} = \varepsilon_0 \chi^{(3)} E_0^3 e^{i(3\omega)t} \text{ Third Harmonic Generation} \]
این جمله قطعی غیر خطی دارای 216 جمله است. در آنها جملاتی شامل هارمونیک‌های سوم هر یک از بسامدها تابش ورودی به محیط غیر خطی و همچنین بسیار بسامدهای تولید شده متفاوت خواهد بود:

\[ \pm \omega_1, \pm \omega_2, \pm \omega_3, \pm 3\omega_1, \pm 3\omega_2, \pm 3\omega_3, (\pm 2\omega_1 \pm \omega_2), (\pm 2\omega_1 \pm \omega_3), (\pm 2\omega_2 \pm \omega_3), (\pm \omega_1 \pm 2\omega_2), (\pm \omega_1 \pm 2\omega_3), (\pm \omega_2 \pm 2\omega_3), (\pm \omega_1 \pm \omega_2 \pm \omega_3), \]

\[ P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(2)} E^2 + \chi^{(3)} E^3 + \ldots \right) = P^{(1)} + P^{(2)} + P^{(3)} \]

\[ E = E_{01} e^{i\omega_1 t} + E_{01}^* e^{-i\omega_1 t} + E_{02} e^{i\omega_2 t} + E_{02}^* e^{-i\omega_2 t} + E_{03} e^{i\omega_3 t} + E_{03}^* e^{-i\omega_3 t} \]

\[ P^{(3)} = \varepsilon_0 \chi^{(3)} E^3 = \varepsilon_0 \chi^{(3)} (E_{01} e^{i\omega_1 t} + E_{02} e^{i\omega_2 t} + E_{03} e^{i\omega_3 t} + c.c.)^3 \]